

Section A: Counting

1 (6 marks)

A password for a computer system is to be constructed from the uppercase letters $\{A, B, \dots, Z\}$ and lowercase letters $\{a, b, \dots, z\}$ of the Roman alphabet and the decimal digits $\{0, 1, \dots, 9\}$.

- (i) [1 mark] What is the cardinality, $n = \#C$, of the set of all allowable password characters, C ?
- (ii) [2 marks] How many passwords, $\#P$, are there in the allowable set of passwords, P , when the password length, r , must be between 8 and 10 characters, with no other restrictions?
- (iii) [2 marks] How many passwords are there when the password must be between 8 and 10 characters in length, with the first character restricted to an uppercase letter and the second character restricted to a decimal digit.
- (iv) [1 mark] Comment on the relative magnitudes of the answers from the previous two parts of this question.

2 (5 marks)

You are required to create a six-digit *personal identity number* (PIN) code, where the six digits are each selected from the decimal digit set $\{0, 1, 2, 3, 4, 5\}$. In each case below, state how many PIN codes there are.

- (i) [1 mark] When the code must be in strictly-decreasing order?
- (ii) [1 mark] With no restrictions on the code?
- (iii) [1 mark] With no decimal digit repeated?
- (iv) [1 mark] With no decimal digit repeated and with the first and last digits being 0 and 5 respectively.
- (v) [1 mark] Where no digit should appear in its naturally-ordered position, where the PIN code for every digit being in its naturally-ordered position is 0, 1, 2, 3, 4, 5.

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3 (5 marks)

You have an unlimited supply of chocolate sweets uniformly distributed over ten different flavours.

- (i) [1 mark] You decide to eat six different flavours. How many different combinations are available for you to eat?
- (ii) [1 mark] Your friend also decides to eat six chocolates, but she wants to have the option of having more than one of any flavour. How many different combinations are available for her to eat?
- (iii) [3 marks] Another friend also decides to eat six chocolates. He decides to select four of his chocolates such that he has a mixture of his two favourite flavours, with at least one from each of these flavours. The other two chocolates are two different flavours from the remaining flavours. How many different combinations are available for him to eat.

4 (5 marks)

Consider the subset of the natural numbers $\mathbb{N}_{100}$, where $\mathbb{N}_{100} = \{1, 2, \dots, 100\}$.

- (i) [1 mark] Determine the cardinality of the following three subsets of $\mathbb{N}_{100}$.
 - A , where A is comprised of all members of $\mathbb{N}_{100}$ that have 3 as a factor.
 - B , where B is comprised of all members of $\mathbb{N}_{100}$ that have 4 as a factor.
 - C , where C is comprised of all members of $\mathbb{N}_{100}$ that have 5 as a factor.
- (ii) [2 marks] Determine the cardinality of all possible intersections of A , B and C .
- (iii) [2 marks] Determine the number of members in the set $\mathbb{N}_{100}$ that have at least one of 3, 4 or 5 as a factor. State the principle that you are using and show your working. How many members of $\mathbb{N}_{100}$ do not have any of these three factors.

Section B: Discrete probability

5 (7 marks)

A fair coin is flipped ten times. Let A be the set of outcomes that contain at least two heads. Let B be the set of outcomes that contain at least three tails.

For these events

- (i) [2 marks] Determine $P(A)$ and $P(B)$.
- (ii) [1 mark] Determine the probability of $P(A \cap B)$.
- (iii) [1 mark] Determine $P(A|B)$.
- (iv) [1 mark] Determine $P(B|A)$.
- (v) [1 mark] Are the events A and B independent? Explain your answer mathematically.
- (vi) [1 mark] Give a *brief* explanation to your answer concerning the independence of events A and B .

6 (3 marks)

A person is known to tell the truth 90% of the time. That person rolls a die and reports to another person, not present at the time, that a maximum score of six was rolled. Define a pair of events A and B and use Bayes' theorem to determine the probability that a six was actually rolled. Show all of your working.

7 (5 marks)

Two players are playing a dice-based game, where higher dice values are better in terms of winning the game. One player always uses a fair die and the other always uses a loaded die.

Suppose that the loaded die has the following probability distribution:

$P(1) = 0.2, P(2) = 0.2, P(3) = 0.2, P(4) = 0.2, P(5) = 0.1, P(6) = 0.1$. Determine:

- (i) [2 marks] The expected value and variance of the loaded die (show your working).
- (ii) [2 marks] The expected value and variance of a fair die (show your working).
- (iii) [1 mark] Which player would you expect to win and why?

Section C: Graphs

8 (9 marks)

This question refers to the graph in Figure 1.

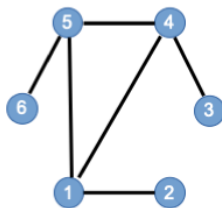


Figure 1: Graph for question 8

- (i) [1 mark] Is it a directed or undirected graph?
- (ii) [1 mark] What is the order of the graph, and the size of the graph?
- (iii) [1 mark] Write down the adjacency matrix of the graph.
- (iv) [1 mark] Write down the degree of each of the vertices in the graph's set of vertices: $V = \{1, 2, \dots, 6\}$. A list in the numerical order of the vertices will suffice.
- (v) [1 mark] Write down the degree sequence of the graph.
- (vi) [1 mark] Write down the distance matrix for the graph with the same row and column ordering as the graph's numerical vertex labels.
- (vii) [1 mark] What is the diameter of the graph?
- (viii) [1 mark] How many cycles does the graph have?
- (ix) [1 mark] Does the graph have a Hamiltonian cycle?

9 (5 marks)

- (i) [1 mark] Write down the *degree sequence* of a complete graph of order n .
- (ii) [1 mark] What is the diameter of a complete graph for a graph of order $n \geq 3$, and what is the diameter when a single edge is removed?
- (iii) [1 mark] How many Hamiltonian paths does the complete graph K_n have, where n is the order of the graph.
- (iv) [2 marks] Determine the smallest *order* of a *complete* graph that has a *size* of at least 100. Show your working.

Section D: Propositional and Predicate Logic

10 (10 marks)

- (i) [7 marks] Which of the following equivalences hold for the NAND $\uparrow$ and NOR $\downarrow$ connectives? Justify your answers: if an equivalence is correct then sketch a proof using algebraic properties; if not, then provide an assignment of truth values to the variables that will make it fail.

(a) [2 marks] $(p \downarrow (q \uparrow r)) \equiv (p \downarrow q) \uparrow (p \downarrow r)$

(b) [2 marks] $\neg(p \downarrow q) \equiv (\neg p) \uparrow (\neg q)$

(c) [3 marks] The associativity over NOR $\downarrow$, clearly state the equivalence as a formula first.

- (ii) [3 marks] Express the **exclusive-or** (*xor*) connective over the two variables p and q , with only $\neg$ and $\vee$. Show the steps involved.

11 (10 marks)

The variables p, q, r have the following meanings:

- p : York is a nice place to live;
- q : I want to live in York;
- r : It is expensive to live in York

- (i) [5 marks]

Consider the following sentences:

If it is expensive to live in York, then if I want to live in York, then York is a nice place to live.

York is not a nice place to live.

Therefore, either it is not expensive to live in York or I don't want to live in York.

(a) [2 marks] Write the sentences as arguments with p, q and r .

(b) [3 marks] Determine the validity of the argument and provide an explanation.

The variables p, q, r have the following meanings (repeated for convenience):

- p : York is a nice place to live;
- q : I want to live in York;
- r : It is expensive to live in York

(ii) [5 marks]

Consider the following sentences:

If York is a nice place to live, then I want to live in York.

If I want to live in York, then it is expensive to live in York.

York is a nice place to live.

Therefore, it is expensive to live in York.

- (a) [2 marks] Write the sentences as arguments with p, q and r .
- (b) [3 marks] Determine the validity of the argument and provide an explanation.

Section E: Sets and Inductive Arguments

12 (10 marks)

- (i) [1 mark] Prove or disprove by giving a counterexample that the product of any three consecutive prime numbers is divisible by 5.
- (ii) [4 marks] Show, by mathematical induction, that for every positive integer n , the sum of all n consecutive positive integers is less than $n(n + 1)$.
- (iii) [3 marks] Given three sets A , B and C , if $A \cap B \cap C \subseteq A \cup B \cup C$ is true then using a translated proposition with the predicates $x \in A$, $x \in B$ and $x \in C$, show that it is a tautology. State clearly the algebraic laws used at each step.
- (iv) [2 marks] If the set $A = \{SOF1, THE1\}$ and the set $S = \{SOF2, THE2, HCI1, SYS1, DAT1, THE3\}$ then.
 - (a) [1 mark] List all the members of the power-set of A .
 - (b) [1 mark] Give a partition P_S of S such that $\#P_S$ (the cardinality of P_S) is equal to 3 and each element of P_S contains at least two elements.

Section F: Functions and Relations

13 (15 marks)

(i) [5 marks]

Let $Bool = \{T, F\}$. Logical connectives with two inputs can be viewed as functions with product source and $Bool$ target. Show the functions for each of the logical connectives $NAND(\uparrow)$ and $Equivalence(\Leftrightarrow)$

(a) [1 mark] as sets of ordered pairs.

(b) [2 marks] as digraphs.

(c) [2 marks] as adjacency matrices - each row should be associated with a distinct pair of logical inputs.

(ii) [7 marks] Let $A = \{SOF1, THE1\}$, $S = \{SOF2, THE2, HCI1, SYS1, DAT1\}$ and R be a relation from A to S defined by the matrix M_R as:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

where the rows are elements from A and the columns are elements from S .

(a) [2 marks] Define R as a set of ordered pairs and draw the corresponding digraph.

(b) [1 mark] Use the digraph to explain if R is a function or not.

(c) [1 mark] Use the digraph to explain if R is total or not.

(d) [1 mark] Use the digraph to explain if R is a total function or not.

(e) [1 mark] Use the digraph to explain if R is an injection or not.

(f) [1 mark] Use the digraph to explain if R is surjective or not.

(iii) [3 marks] Let G be the $>$ (greater than) relation, and E the $=$ (equal to) relation, on the set $A = \{3, 5, 7, 8, 9\}$. Use the matrices of the two relations (G and E) to show that, $G \cup E$ is indeed the $\geq$ (greater than or equal to) relation.

Section G: Ordering and Equivalence

14 (5 marks)

Let R be the $\geq$ relation on the set $A = \{3, 5, 7, 8, 9\}$.

- (i) [2 marks] Determine the digraph of the $\geq$ relation.
- (ii) [3 marks] Determine the Hasse diagram of the $\geq$ relation.